PERGAMON

International Journal of Solids and Structures 37 (2000) 7029–7042

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Computing exact, elastodynamic linear three-dimensional solutions for plates from classical two-dimensional solutions

P. Ladevèze [a,*], Z. Qian [b], J.G. Simmonds [b]

[a] *Laboratoire de Mécanique et Technologie, Ecole Normale Supérieur de Cachan (CNRS-Paris VI), 61, Avenue du President Wilson, 94235 Cachan Cedex, France*

[b] *Department of Civil Engineering, University of Virginia, Charlottesville, VA, USA*

Received 10 November 1999

Abstract

Given an approximate time-dependent distribution of midplane vertical displacement and three-dimensional transverse shear and normal stresses in a platelike elastic body undergoing flexure – the quantities delivered by the Kirchhoff (classical) theory – we construct *exact* solutions of the equations of motion of linear three-dimensional elasticity. This is accomplished by (1) solving an auxiliary *spatially hyperbolic* system of partial differential equations (in which time enters only parametrically) and (2) choosing residual body and surface forces and initial conditions to insure satisfaction of all three-dimensional field equations, boundary, and initial conditions. The residual quantities which, in general, are significant only near the edges of the plate, serve as meaningful physical measures of the errors in classical plate theory. The special difficulties posed by plates with sharp corners are mentioned, but are left for future treatment.

Keywords: Elasticity; Plate theory

1. Introduction

Although refinements and extensions of the Prager–Synge (1947) hypercircle method have been applied successfully to assess the errors in the three-dimensional stress fields that one can infer from the Kirchhoff (classical) *static* theory of plates (see Ladevèze, 1985, where relevant work may be found), little has been done on how to estimate errors in *dynamics*. Such error assessments are important, because, as is well known, dynamic phenomena can produce entirely new effects not seen in statics (resonant vibrations, stress reversals due to reflections, wave amplification due to special geometries).

Ladevèze and Simmonds (1996) considered the similar but simpler problem of estimating errors in the solutions of the equations of motion of the elementary (Euler–Bernoulli) theory of beams of narrow rectangular cross-section as compared to solutions of the "exact" equations of elastodynamic plane stress

* Corresponding author. Tel.: +33-1-47-40-22-38; fax: +33-1-47-40-22-40.
E-mail address: ladeveze@lmt.ens-cachan.fr (P. Ladevèze).

0020-7683/00/$ - see front matter
PII: S0020-7683(99)00326-1

theory. Also, Qian and Simmonds (1998) showed how to compute the pointwise errors made in three-dimensional elastodynamic solutions inferred from solutions of the classical linear elastodynamic theory of *axisymmetric* homogeneous isotropic plates. The idea, which we adopt here for homogeneous, elastically isotropic plates of *general* shape subject to normal surface loads and fixed to a rigid boundary, though straightforward, requires careful attention to mathematical details. Nevertheless, it has a direct physical appeal: one takes the approximate time-dependent three-dimensional stress and displacement fields to be those inferred from Kirchhoff theory and then asks, "What residual external loads and initial conditions must be added to the exact, three-dimensional equations of elasticity to make the approximate fields exact?". Next, we show, via an integral representation using a Green's function, that these residual quantities may be computed explicitly in terms of solutions of the Kirchhoff plate equations, although, in practice, the order of magnitude estimates may suffice. Finally, we show that our analytic results can be simplified considerably if we introduce asymptotic expansions in a small parameter representing the thickness-to-diameter ratio of the plate. Obviously, our approach includes static problems, although no one to our knowledge has applied it even to this special case.

In summary, given an initial/boundary-value problem on a plate-like domain in three-dimensional Euclidean space, we solve exactly a "neighboring" problem. The increments so introduced in the data, which are computable *explicitly*, then serve as a measure of the "distance" between the two problems. As the data in a plate problem are rarely known exactly, the perturbed data may be sufficiently close to the original data to be acceptable in practice. If not, our analysis will show where Kirchhoff theory must be refined. (Thus, we do not attempt to specify a precise norm to measure "distance". Rather, we present an algorithm for computing changes in loads and initial conditions and leave it to engineering judgment as to whether, in specific cases, these are tolerable.)

To concentrate on essentials, we consider a simply connected plate, clamped along a smooth boundary and subject to initial and boundary conditions, external surface and body forces such that the plate is in a state of pure bending. Moreover, in the associated two-dimensional Kirchhoff plate equations, the initial conditions are taken to be zero and the only given external force is a normal pressure that may vary in space and time.

The original contributions of the present article are (1) a technique for constructing *exact*, explicit three-dimensional elastodynamic solutions for *arbitrary* plate-like bodies (excluding sharp corners), which starts from two-dimensional solutions of Kirchhoff plate theory; (2) the observation that the residual loads and initial conditions introduced to solve the three-dimensional problem can serve as a measure of the accuracy of the underlying approximate two-dimensional plate theory.

2. The governing equations

In an inertial right-handed Cartesian reference frame $Oxyz$ with the standard set of fixed, orthonormal basis vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, particles in the interior of the undeformed plate have the representation

$$\mathscr{P} : \mathbf{x} = (\mathbf{r}, z), \quad \mathbf{r} = (x, y) \in \Omega, \qquad z \in (-H, H), \tag{2.1}$$

where Ω is the *midplane* of the plate and $2H$ is its thickness. We shall assume that the boundary of Ω is smooth and has the parametric representation

$$\partial\Omega : \mathbf{r} = \hat{\mathbf{r}}(s) = \hat{x}(s)\mathbf{i} + \hat{y}(s)\mathbf{j}, \quad s \in [0, L], \tag{2.2}$$

where s is the arc length.

Denoting the in-plane, transverse, and normal components of the symmetric stress tensor by σ_{xx}, σ_{xy}, σ_{yy}, τ_x, τ_y, and σ, respectively, and the in-plane and normal components of the displacement vector by U_x, U_y, and W, respectively, we have, from the classical three-dimensional linear theory of isotropic elasticity (Timoshenko and Goodier, 1970), the six displacement–stress relations

$$G(U_{x,z}+W_{,x})=\tau_x,\qquad G(U_{y,z}+W_{,y})=\tau_y,\qquad EW_{,z}=\sigma-\nu(\sigma_{xx}+\sigma_{yy}),\tag{2.3}$$

$$G(U_{x,y}+U_{y,x})=\sigma_{xy},\qquad EU_{x,x}=\sigma_{xx}-\nu(\sigma_{yy}+\sigma),\qquad EU_{y,y}=\sigma_{yy}-\nu(\sigma_{xx}+\sigma),\tag{2.4}$$

and the three equations of motion

$$\tau_{x,x}+\tau_{y,y}+\sigma_{,z}+f_z=\rho W_{,tt},\tag{2.5}$$

$$\sigma_{xx,x}+\sigma_{xy,y}+\tau_{x,z}+f_x=\rho U_{x,tt},\qquad \sigma_{xy,x}+\sigma_{yy,y}+\tau_{y,z}+f_y=\rho U_{y,tt}.\tag{2.6}$$

Here, G is the shear modulus, ν is Poisson's ratio, $E=2(1+\nu)G$ is Young's modulus, f_x, f_y, and f_z are body forces, ρ is the mass density, and a comma followed by a subscript denotes differentiation with respect to the subscript.

Pure bending means that

$$\left\{\begin{array}{c}\tau_x,\tau_y,W\\ \sigma_{xx},\sigma_{xy},\sigma_{yy},\sigma,U_x,U_y\end{array}\right\}\text{ are }\left\{\begin{array}{c}\text{even}\\ \text{odd}\end{array}\right\}\text{ functions of } z.\tag{2.7}$$

In view of Eq. (2.7), the boundary conditions on the faces of the plate may be expressed as

$$\{\sigma,\tau_x,\tau_y\}(\mathbf{r},\pm H,t)=\tfrac{1}{2}\{\pm p,q_x,q_y\}(\mathbf{r},t),\quad \mathbf{r}\in\Omega,\ 0<t,\tag{2.8}$$

while on the edge of the plate we have

$$\{U_x,U_y,W\}(\mathbf{r},z,t)=0,\quad \mathbf{r}\in\partial\Omega,\ z\in[-H,H],\quad 0<t.\tag{2.9}$$

To complete the specification of the three-dimensional problem, we have the initial conditions

$$\{U_x,U_y,W,U_{x,t},U_{y,t},W_{,t}\}(\mathbf{r},z,0)=\{U_x^0,U_y^0,W^0,\dot U_x^0,\dot U_y^0,\dot W^0\}(\mathbf{r},z),\quad \mathbf{r}\in\Omega,\ z\in[-H,H],\tag{2.10}$$

where $U_x^0,\dot U_x^0$, etc., are given functions.

For simplicity, *we shall assume that in the original problem, the external surface shears, the body forces, and the initial conditions vanish.*

The associated Kirchhoff plate equation, from whose solutions we shall built our exact three-dimensional solutions, are from Timoshenko and Woinowsky-Krieger (1959),

$$-D\Delta\Delta w+p(\mathbf{r},t)=mw_{,tt},\quad \mathbf{r}\in\Omega,\ 0<t,\tag{2.11}$$

where $D=2EH^3/3(1-\nu^2)$ is the bending stiffness, w is the midplane vertical displacement, Δ is the two-dimensional Laplacian, and $m=2H\rho$ is the mass/area. The boundary conditions for the Kirchhoff plate equations are

$$\{w,w_{,n}\}(\mathbf{r},t)=0,\quad \mathbf{r}\in\partial\Omega,\ 0<t,\tag{2.12}$$

where $w_{,n}$ is the outward normal derivative at the boundary; the initial conditions are

$$\{w,w_{,t}\}(\mathbf{r},0)=0,\quad \mathbf{r}\in\Omega.\tag{2.13}$$

Our task is to choose the surface shears, body forces, and three-dimensional initial conditions – the *residual* loads and initial conditions, as we shall call them – so that we may construct exact explicit solutions of the resulting three-dimensional field equations.

To simplify the notation, we set

$$\overline{f}=\frac{f}{E},\qquad \alpha=\frac{E}{G}=2(1+\nu),\qquad c^2=\frac{\nu}{1-\nu}.\tag{2.14}$$

We then use the three displacement–stress relations given by Eq. (2.4), re-written in the form

$$\alpha\overline{\sigma}_{xy} = U_{x,y} + U_{y,x}\,,$$
$$(1-v^2)\overline{\sigma}_{xx} = (U_{x,x} + vU_{y,y}) + v(1+v)\overline{\sigma}, (1-v^2)\overline{\sigma}_{yy} = (U_{y,y} + vU_{x,x}) + v(1+v)\overline{\sigma} \tag{2.15}$$

to eliminate $\overline{\sigma}_{xy}, \overline{\sigma}_{xx}$, and $\overline{\sigma}_{yy}$ in favor of U_x, U_y, W, and $\overline{\sigma}$. Using Eq. (2.14) and substituting the above expressions into Eq. (2.3), we obtain

$$U_{x,z} + W_{,x} = \alpha\overline{\tau}_x, \qquad U_{y,z} + W_{,y} = \alpha\overline{\tau}_y,$$
$$v(U_{x,x} + U_{y,y}) + (1-v)W_{,z} = (1+v)(1-2v)\overline{\sigma}. \tag{2.16}$$

We now assume that the right-hand sides of Eq. (2.16) are *given* and satisfy the upper and lower face boundary conditions (2.8). (Ultimately, $\overline{\tau}_x, \overline{\tau}_y$ and $\overline{\sigma}$ are to be expressed in terms of solutions of the Kirchhoff plate equations.) Thus, we are left with a *hyperbolic system* of partial differential equations for the two unknowns U and W as functions of the two spatial variables x and y, with time t appearing only parametrically. The solution of this system, subject to certain conditions on the midplane and edges of the three-dimensional plate, is going to yield the desired exact three-dimensional elasticity solutions. Being hyperbolic, the system is amenable to solution by certain (modified) classical techniques, as we now show.

3. Reduction of the governing equations

First, we assume that the external surface shears may be expressed in terms of a *residual* load potential as

$$\overline{q}_x = \phi_{,x}(\mathbf{r}, t), \qquad \overline{q}_y = \phi_{,y}(\mathbf{r}, t), \tag{3.1}$$

where

$$\phi(\mathbf{r}, t) = 0, \quad \mathbf{r} \in \partial\Omega, \ \ 0 < t. \tag{3.2}$$

(This assumption proves to place no limitation on our approach.) Next, we introduce the change of variables

$$W = \tilde{W} + \tfrac{1}{2}\alpha\phi, \qquad \overline{\tau}_x = \tilde{\tau}_x + \tfrac{1}{2}\phi_{,x}\,, \qquad \overline{\tau}_y = \tilde{\tau}_y + \tfrac{1}{2}\phi_{,y}\,. \tag{3.3}$$

Then, using Eqs. (2.14), (2.15), (3.1), and (3.2), we may re-write our field Eqs. (2.5), (2.6), and (2.16) in the forms

$$\overline{f}_z = \overline{\rho}(\tilde{W} + \tfrac{1}{2}\alpha\phi)_{,tt} - (\tilde{\tau}_{x,x} + \tilde{\tau}_{y,y}) - \overline{\sigma}_{,z} - \tfrac{1}{2}\Delta\phi, \tag{3.4}$$

$$\overline{f}_x = \overline{\rho}U_{x,tt} - \tilde{\tau}_{x,z} - \tfrac{1}{2}(1-v^2)^{-1}[2U_{x,xx} + (1-v)U_{x,yy} + (1+v)U_{y,xy} + 2v(1+v)\overline{\sigma}_{,x}], \tag{3.5}$$

$$\overline{f}_y = \overline{\rho}U_{y,tt} - \tilde{\tau}_{y,z} - \tfrac{1}{2}(1-v^2)^{-1}[2U_{y,yy} + (1-v)U_{y,xx} + (1+v)U_{x,xy} + 2v(1+v)\overline{\sigma}_{,y}], \tag{3.6}$$

$$U_{x,z} = -\tilde{W}_{,x} + \alpha\tilde{\tau}_x, \qquad U_{y,z} = -\tilde{W}_{,y} + \alpha\tilde{\tau}_y, \tag{3.7}$$

$$v(U_{x,x} + U_{y,y}) + (1-v)\tilde{W}_{,z} = (1+v)(1-2v)\overline{\sigma}. \tag{3.8}$$

By Eqs. (3.2), (3.3), and (3.7), we may re-write our boundary conditions (2.9) as

$$\{\tilde{W}_{,x}, \tilde{W}_{,y}, \tilde{W}\}(\mathbf{r}, z, t) = \alpha\{\tilde{\tau}_x, \tilde{\tau}_y, 0\}(\boldsymbol{v}, z, t), \quad \mathbf{r} \in \partial\Omega, \ \ z \in [-H, H], \ \ 0 < t. \tag{3.9}$$

The homogeneous initial conditions on the in-plane displacements remain unchanged while those on the normal displacement become

$$\{\tilde{W}, \tilde{W}_{,t}\} = -\tfrac{1}{2}\alpha\{\phi, \phi_{,t}\}(\mathbf{r}, 0). \tag{3.10}$$

Note that the boundary conditions (2.8), which now take the form,

$$\{\overline{\sigma}, \tilde{\tau}_x, \tilde{\tau}_y\} = \tfrac{1}{2}\{\pm\overline{p}, 0, 0\}(\mathbf{r}, t), \quad \mathbf{r} \in \Omega, \ 0 < t, \tag{3.11}$$

are to be satisfied by the modified fields $\tilde{\tau}_x$, $\tilde{\tau}_y$, and $\overline{\sigma}$.

To reduce our field equations further, we assume that $0 < \nu < \frac{1}{2}$ and differentiate both sides of Eq. (3.8) with respect to z (making obvious assumptions on smoothness, as we do throughout the article). We then eliminate $U_{x,xz}$ and $U_{y,yz}$ from the resulting equation by using the first term in Eq. (3.7), differentiated with respect to x, and the second term in Eq. (3.7), differentiated with respect to y. These manipulations yield the non-homogeneous *spatial wave equation*

$$\begin{aligned}\tilde{W}_{,zz} - c^2\Delta\tilde{W} &= \left(\frac{1+\nu}{1-\nu}\right)[(1-2\nu)\overline{\sigma}_{,z} - 2\nu(\tilde{\tau}_{x,x} + \tilde{\tau}_{y,y})] \equiv f(\mathbf{r}, z, t), \quad r \in \Omega,\\ & z \in (-H, H), \ 0 < t,\end{aligned} \tag{3.12}$$

where the "wave speed" c in the z-direction is given by the third term in Eq. (2.14). Note that time enters Eq. (3.12) only as a parameter and, therefore, will be suppressed henceforth as an argument except in Appendix A.

If we now identify $W(\mathbf{r}, 0)$ with the vertical deflection $w(\mathbf{r})$ of the Kirchhoff plate theory, then, because W is an *even* function of z (as is $\tilde{W}$), our problem is reduced to the following: solve Eq. (3.12) on the open domain $\Omega \times (0, H)$, subject to the "initial" conditions [1]

$$\tilde{W}(\mathbf{r}, 0) = w(\mathbf{r}) - \tfrac{1}{2}\alpha\phi(\mathbf{r}) \equiv \chi(\mathbf{r}), \quad \tilde{W}_{,z}(\mathbf{r}, 0) = 0, \qquad \mathbf{r} \in \Omega \tag{3.13}$$

and the boundary conditions (3.9). In general, such boundary conditions (which are equivalent to specifying both $\tilde{W}$ *and* its normal derivative $\tilde{W}_{,n}$) *over-determine* the solution of the two-dimensional wave equation and, therefore, must satisfy a *consistency condition* – a condition that will ultimately determine ϕ and hence the body forces from the right-hand sides of Eqs. (3.4)–(3.6). The residual initial conditions we spoke of will be determined from Eq. (2.10).

4. Solution for $\tilde{W}$

We shall follow a standard procedure in linear partial differential equations and attempt to satisfy Eq. (3.12) and the conditions (3.13) with the aid of a *free-space Green's function* $G(\mathbf{x}; \overline{\mathbf{x}})$ (See, e.g. Copson, 1975). This will lead, after some mathematical maneuvers to work around singular behavior on characteristic cones, to an integral representation for $\tilde{W}$. Satisfaction of the (two, over-specified) boundary conditions (3.13) leads, finally, to an integral equation for the unknown residual load potential ϕ.

We require our Green's function to satisfy the (distributional) differential equation

$$G_{,zz} - c^2\Delta G = \delta(\mathbf{x} - \overline{\mathbf{x}}), \quad \mathbf{x}, \overline{\mathbf{x}} \in \Omega \times (0, H) \tag{4.1}$$

and the "causal" condition

$$G(\overline{\mathbf{x}} - \mathbf{x}) = 0, \quad z > \overline{z}, \tag{4.2}$$

where δ is the delta distribution. The Green's function has the well-known form (Copson, 1975, p. 107),

$$2\pi c G(\overline{\mathbf{x}} - \mathbf{x}) = \begin{cases} [c^2(\overline{z} - z)^2 - |\,\overline{\mathbf{r}} - \mathbf{r}\,|^2]^{-1/2}, & c(\overline{z} - z) > |\,\overline{\mathbf{r}} - \mathbf{r}\,|, \\ 0, & c(\overline{z} - z) < |\,\overline{\mathbf{r}} - \mathbf{r}\,|. \end{cases} \tag{4.3}$$

[1] Remember, we are suppressing the argument t.

Thus, G vanishes outside of and is singular on the boundary of the (open, solid, one-sheeted) *characteristic cone*

$$C(\bar{\mathbf{x}}) = \{\mathbf{x} \,|\, c(\bar{z} - z) > |\bar{\mathbf{r}} - \mathbf{r}|, \bar{z} \leqslant H\}. \tag{4.4}$$

If we attempt to obtain an integral representation for $\tilde{W}(\bar{\mathbf{x}})$ in the standard way by using integration by parts and the divergence theorem to remove derivatives on G and put them on $\tilde{W}$, we encounter divergent integrals along the boundary of the characteristic cone. Such obstacles were overcome by Marcel Riesz who introduced a fractional integral operator and performed analytic continuation on the order of the operator. (A very readable exposition may be found in Copson (1975).) Fortunately, there is a much simpler way in our problem: we work with the (slightly) *shifted* Green's function $G(\bar{\mathbf{r}} - \mathbf{r}, \bar{z} + \varepsilon - z), \varepsilon > 0$, which is infinitely differentiable on and in a neighborhood of C. Then, after suitable manipulations, we let $\varepsilon \to 0$. The details are as follows.

First, to simplify the notation and to minimize minus signs, we make a change of variables and introduce a parameter:

$$\mathbf{r} - \bar{\mathbf{r}} = H\boldsymbol{\rho}, \qquad c(\bar{z} - z) = H\zeta, \quad \rho = |\boldsymbol{\rho}|, \quad \bar{\zeta} = c\bar{z}/H. \tag{4.5}$$

Further, as explained above, we shift the location of the delta in Eq. (4.1) to the point $(\bar{\mathbf{r}}, \bar{z} + \varepsilon)$ and denote cH times the corresponding Green's function and the characteristic cone by $\tilde{G}(\rho, \zeta)$ and $C(\bar{\zeta})$, respectively. Thus, in a local system of circular cylindrical coordinates (ρ, θ, ζ), Eqs. (3.12) and (4.1) may be given the respective forms

$$\begin{aligned}(\rho\tilde{W}_{,\zeta})_{,\zeta} - (\rho\tilde{W}_{,\rho})_{,\rho} - (\rho^{-1}\tilde{W}_{,\theta})_{,\theta} &= (H/c)^2 \rho f(\bar{\mathbf{r}} + H\boldsymbol{\rho}, \bar{z} - H\zeta/c) \\ &\equiv \rho\tilde{f}(\rho, \theta, \zeta),\end{aligned} \tag{4.6}$$

$$(\rho\tilde{G}_{,\zeta})_{,\zeta} - (\rho\tilde{G}_{,\rho})_{,\rho} - (\rho^{-1}\tilde{G}_{,\theta})_{\theta} = \delta(\rho, \zeta + \varepsilon)/2\pi. \tag{4.7}$$

We now multiply Eq. (4.6) by $\tilde{G}$, subtract $\tilde{W}$ multiplied by Eq. (4.7), and integrate over the intersection of the plate $\mathscr{P}$ and the (transformed) solid characteristic cone $C(\bar{\zeta})$. Next, we apply the divergence theorem in the coordinate system (ρ, θ, ζ). Then, along the generators of ∂C, the boundary of C, we set $\rho = \zeta = \sigma/\sqrt{2}$ and note that, on these generators, $\sqrt{2}f_{,\sigma} = f_{,\rho} + f_{,\zeta}$. Finally, we let $\rho = \hat{\rho}(\theta)$, $\zeta = \bar{\zeta}$ represent the boundary of the intersection of C with the midplane Ω of the plate, as in Fig. 1; if C and $\partial\Omega$ do not intersect, then $\hat{\rho} = \bar{\zeta}$.

Let AB, CD, and DA, as in Fig. 1, denote the two (and possibly three) straight line segments which form the boundary of the intersection of the first quadrant of the $\rho\zeta$-plane with $C(\bar{\zeta})$. On DA (if it is not empty), the boundary conditions (3.9) imply that

$$\tilde{W} = 0 \quad \text{and} \quad \tilde{W}_{,n} = \alpha\tilde{\tau}_s, \tag{4.8}$$

where $\tilde{W}_{,n}$ is the directional derivative of $\tilde{W}$ along the outward normal to the edge and

$$\tilde{\tau}_s = \tilde{\tau}_x(s, z)\hat{y}'(s) - \tilde{\tau}_y(s, z)\hat{x}'(s). \tag{4.9}$$

Noting the "initial" conditions (3.13), we have, altogether

$$\int_0^{2\pi} (I_{AB} + I_{CD} + I_{DA})\,\mathrm{d}\theta = \int_0^{2\pi} \int_0^{\hat{\rho}(\theta)} \rho \int_\rho^{\bar{\zeta}} \tilde{G}(\rho, \zeta)\tilde{f}(\rho, \theta, \zeta)\,\mathrm{d}\zeta\,\mathrm{d}\rho\,\mathrm{d}\theta, \tag{4.10}$$

where

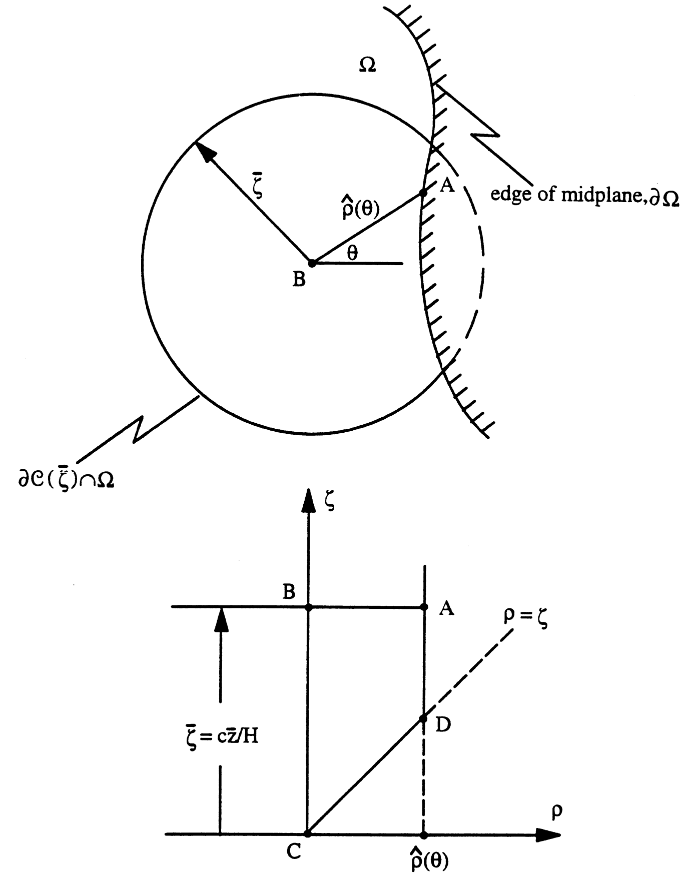

Fig. 1. Geometry of the intersection of the characteristic cone $C(\bar{\zeta})$ with the edge and midplane of the plate.

$$I_{AB} = -\int_0^{\hat{\rho}(\theta)} \chi(\tilde{G}_{,\zeta})_{AB}\rho\,\mathrm{d}\rho, \tag{4.11}$$

$$I_{CD} = (1/\sqrt{2})\int_0^{\sqrt{2}\hat{\rho}(\theta)} (\tilde{W}\tilde{G}_{,\sigma} - \tilde{G}\tilde{W}_{,\sigma})_{CD}\sigma\,\mathrm{d}\sigma, \tag{4.12}$$

$$I_{DA} = -\alpha\sqrt{\hat{\rho}^2(\theta) + \hat{\rho}'^2(\theta)}\int_{\hat{\rho}(\theta)}^{\bar{\zeta}} (\tilde{G}\tilde{\tau}_s)_{DA}\,\mathrm{d}\zeta, \tag{4.13}$$

and where the notation $(\cdot)_{AB}$ means that the quantity in parentheses is evaluated on AB.

We now consider each of these integrals as $\varepsilon \to 0$, assuming that $\tilde{W}$ and its first partial derivatives are continuous on the closure of the region $\Omega \times (-H, H)$ which defines the interior of the undeformed plate.

First, on AB,

$$2\pi\rho\tilde{G}_{,\zeta} = -\rho[(\bar{\zeta}+\varepsilon)^2 - \rho^2]^{-3/2}(\bar{\zeta}+\varepsilon) = -2\pi(\bar{\zeta}+\varepsilon)\tilde{G}_{,\rho}\,. \tag{4.14}$$

Hence, substituting this expression into Eq. (4.11), integrating by parts, and noting that $\chi = 0$ on $\partial\Omega$, we have

$$I_{AB} = -(\tfrac{1}{2}\pi)(\chi)_B - (\overline{\zeta} + \varepsilon)\int_0^{\hat{\rho}(\theta)} \chi,_\rho (\tilde{G})_{AB}\,\mathrm{d}\rho + \begin{cases} 0 & \text{if } \hat{\rho} < \overline{\zeta}, \\ (\overline{\zeta}+\varepsilon)(\chi\tilde{G})_A & \text{if } \hat{\rho} = \overline{\zeta}, \end{cases} \tag{4.15}$$

where the notation $(\cdot)_A$ means that the quantity in parentheses is evaluated at the point A.

Next, on CD,

$$2\pi\ \tilde{G} = (\sqrt{2}\sigma\varepsilon + \varepsilon^2)^{-1/2} \tag{4.16}$$

so that

$$(\tilde{W}\tilde{G},_\sigma - \tilde{G}\tilde{W},_\sigma)\sigma = \sqrt{2}\varepsilon\tilde{G}\tilde{W},_\sigma - [(\sigma + \sqrt{2}\varepsilon)\tilde{G}\tilde{W}],_\sigma. \tag{4.17}$$

Thus, after integration by parts, we have

$$I_{CD} = (1/2\pi)(\tilde{W})_C + \varepsilon\int_0^{\sqrt{2}\hat{\rho}(\theta)} (\tilde{G}\tilde{W},_\sigma)_{CD}\,\mathrm{d}\sigma - \begin{cases} 0 & \text{if } \hat{\rho} < \overline{\zeta}, \\ (\overline{\zeta}+\varepsilon)(\chi\tilde{G})_A & \text{if } \hat{\rho} = \overline{\zeta}. \end{cases} \tag{4.18}$$

We now add Eqs. (4.15) and (4.18) and let $\varepsilon \to 0$. In Eq. (4.15), the limiting form of the integral is improper if the upper limit is $\overline{\zeta}$, because $(G)_{AB}$ then has an inverse square root singularity. However, the singularity is integrable. In Eq. (4.18), $\varepsilon(\tilde{G})_{CD} \to 0$ uniformly as $\varepsilon \to 0$, whereas $|(\tilde{W},_\sigma)_{CD}|$ is bounded by assumption. Thus, in the limit,

$$I_{AB} + I_{CD} = (1/2\pi)[(\tilde{W})_C - (\chi)_B] - \overline{\zeta}\int_0^{\hat{\rho}(\theta)} \chi,_\rho (G)_{AB}\,\mathrm{d}\rho. \tag{4.19}$$

As for I_{DA}, note that

$$2\pi(G)_{DA} = [\zeta^2 - \hat{\rho}^2(\theta)]^{-1/2} \tag{4.20}$$

so that, in the limit as $\varepsilon \to 0, I_{DA}$ has an (integrable) square root singularity at the lower limit of integration, $\zeta = \hat{\rho}(\theta)$.

Finally, substituting Eq. (4.19) and the limiting form of I_{DA} into Eq. (4.10) and solving for $(\tilde{W})_C$, we have

$$\begin{aligned}(\tilde{W})_C = (\chi)_B + \int_0^{2\pi} \Bigg\{ \overline{\zeta}\int_0^{\hat{\rho}(\theta)} \left[\chi,_\rho (G)_{AB} + \rho\int_\rho^{\overline{\zeta}} G(\rho,\zeta)\tilde{f}(\rho,\theta,\zeta)\,\mathrm{d}\zeta\right]\mathrm{d}\rho \\ - \alpha\sqrt{\hat{\rho}^2(\theta) + \hat{\rho}'^2(\theta)}\int_{\hat{\rho}(\theta)}^{\overline{\zeta}} (G\tilde{\tau}_s)_{DA}\,\mathrm{d}\zeta\Bigg\}\,\mathrm{d}\theta.\end{aligned} \tag{4.21}$$

5. Solution for the residual load potential ϕ

To proceed further, we assume that the *curvature* $\kappa(s)$ at each point of $\partial\Omega$ satisfies

$$cH\kappa < 1. \tag{5.1}$$

This assumption eliminates sharp corners (which, as can be inferred from Figs. 1 and 2, would require a special treatment that we do not attempt here) and simplifies the geometry when the characteristic cone intersects the edge of the plate.

It is well known that solutions of the linear three-dimensional plate equations exhibit a rapidly varying edge (or boundary) layer of width O(H) superimposed on a slowly varying interior component. This suggests that we attempt to determine ϕ on two disjoint regions of the midplane, the *edge zone*

$$\Omega_\mathrm{e} \equiv \Omega \cap C(\overline{\mathbf{x}}), \quad \overline{\mathbf{x}} \in \partial\Omega \times [0, H], \tag{5.2}$$

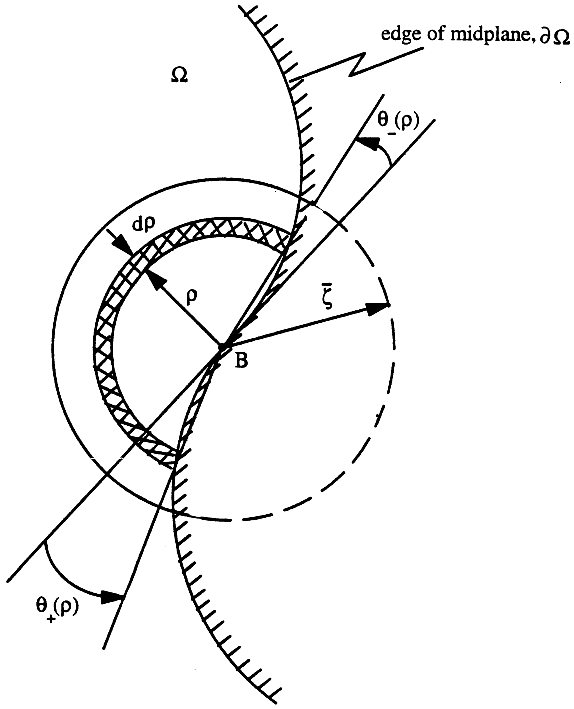

Fig. 2. Geometry of the intersection of the characteristic cone $C(\bar{\zeta})$ with the midplane of the plate when the cone's axis lies on the plate's edge.

that is, the set of points in the midplane that lie in the domain of dependence of the edge, and the *interior*

$$\Omega_\mathrm{i} \equiv \Omega - \Omega_\mathrm{e}. \tag{5.3}$$

First, we choose $\phi|_{\Omega_\mathrm{i}} = 0$ so that $\chi|_{\Omega_\mathrm{i}} = w$. Furthermore, in the interior, $\hat{\rho}(\theta) = \bar{\zeta}$. Thus, in the local coordinate system (4.5), we obtain from Eq. (4.21), the *explicit* formula

$$\tilde{W}(\mathbf{0}, 0) = w(\mathbf{0}) + \frac{1}{2\pi} \int_0^{2\pi} \left\{ \bar{\zeta} \int_0^{\bar{\zeta}} \left[\frac{w_{,\rho}(\rho, \theta)}{\sqrt{\bar{\zeta}^2 - \rho^2}} + \rho \int_\rho^{\bar{\zeta}} \frac{\tilde{f}(\rho, \theta, \zeta)\,\mathrm{d}\zeta}{\sqrt{\zeta^2 - \rho^2}} \right] \mathrm{d}\rho \right\} \mathrm{d}\theta. \tag{5.4}$$

On Ω_e, we must choose ϕ so that $\tilde{W}$ vanishes on the edge of the plate. To do so, we now place the origin B of our local circular cylindrical coordinates (ρ, θ, ζ) on $\partial\Omega$. Furthermore, we now describe the intersection of the characteristic cone $C(\rho)$ with the edge $\partial\Omega$ by specifying the two functions

$$\theta = \pi + \theta_+(\rho) \quad \text{and} \quad \theta = \theta_-(\rho), \qquad 0 \leqslant \rho \leqslant \bar{\zeta}, \tag{5.5}$$

as shown in Fig. 2, and change orders and variables of integration. Finally, we assume that in a neighborhood of any point $B \in \partial\Omega$, we can express the dimensionless directed distance σ from B along $\partial\Omega$ in the form

$$\sigma = \begin{cases} \sigma_+(\rho) & \text{if } \sigma > 0, \\ -\sigma_-(\rho) & \text{if } \sigma < 0. \end{cases} \tag{5.6}$$

Thus, because $\chi = 0$ on $\partial\Omega$, we obtain from Eq. (4.21) the following integral equation or *consistency condition* for the residual load potential:

$$\int_0^{\overline{\zeta}} \frac{\psi(\rho)\,\mathrm{d}\rho}{\sqrt{\overline{\zeta}^2-\rho^2}} = h(\overline{\zeta}), \quad 0 \leqslant \overline{\zeta} \leqslant c, \tag{5.7}$$

where

$$\psi(\rho) = \int_{\theta_-(\rho)}^{\pi+\theta_+(\rho)} \chi_{,\rho}(\rho,\theta)\,\mathrm{d}\theta, \quad 0 \leqslant \rho \leqslant \overline{\zeta}, \tag{5.8}$$

$$\begin{aligned} h(\overline{\zeta}) = {} & \alpha \int_0^{\overline{\zeta}} \int_\rho^{\overline{\zeta}} \left[\frac{\sigma'_-(\rho)\tilde{\tau}_s(-\sigma_-(\rho),\zeta) + \sigma'_+(\rho)\,\tilde{\tau}_s(\sigma_+(\rho),\zeta)}{\sqrt{\zeta^2-\rho^2}} \right] \mathrm{d}\zeta\,\mathrm{d}\rho \\ & - \int_0^{\overline{\zeta}} \int_\rho^{\overline{\zeta}} \frac{1}{\sqrt{\zeta^2-\rho^2}} \int_{\theta_-(\rho)}^{\pi+\theta_+(\rho)} \tilde{f}(\rho,\theta,\zeta)\,\mathrm{d}\theta\,\mathrm{d}\zeta\,\mathrm{d}\rho, \quad 0 \leqslant \overline{\zeta} \leqslant c \end{aligned} \tag{5.9}$$

is a known function.

The left-hand side of Eq. (5.7) is a convolution whose solution, via Laplace transforms, is

$$\psi(\rho) = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\int_0^\rho \frac{h(\xi)\xi\,\mathrm{d}\xi}{\sqrt{\rho^2-\xi^2}}\right]. \tag{5.10}$$

But note that, because χ vanishes on $\partial\Omega$, the right-hand side of Eq. (5.8) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{\theta_-(\rho)}^{\pi+\theta_+(\rho)} \chi(\rho,\theta)\,\mathrm{d}\theta, \quad 0 \leqslant \rho \leqslant \overline{\zeta} \tag{5.11}$$

and hence, to within an unknown constant which must be set to zero because $\chi(0, \theta) = 0$, we are left with the following integral equation to solve:

$$\int_{\theta_-(\rho)}^{\pi+\theta_+(\rho)} \chi(\rho,\theta)\,\mathrm{d}\theta = \frac{2}{\pi}\int_0^\rho \frac{h(\xi)\,\mathrm{d}\xi}{\sqrt{\rho^2-\xi^2}} \equiv g(\rho), \quad 0 \leqslant \rho \leqslant \overline{\zeta}, \tag{5.12}$$

where g is a known function (computable, but perhaps, only numerically).

6. The thin plate approximation

In general, the solution of Eq. (5.12) for χ must be obtained numerically. However, if we assume that the plate is *thin* in the sense that

$$\varepsilon = H/L \ll 1, \tag{6.1}$$

then we can make further progress analytically.

As mentioned earlier, approximate solutions in the three-dimensional theory of elastically isotropic clamped plates exhibit boundary layers of $\mathrm{O}(H)$ along smooth edges. (The behavior near corners, which is more complicated, awaits analysis.) Within such boundary layers, solutions vary rapidly with respect to distance normal to the edge, but generally, more slowly in a direction parallel to the edge. To make our basic integral Eq. (5.12) better reflect this behavior and with an eye toward an approximate perturbation solution, we now introduce a system of dimensionless *edge-zone geodesic coordinates* (σ, τ) by setting

$$s = \bar{s} + H\sigma, \tag{6.2}$$

$$\mathbf{r} = \hat{\mathbf{r}}(\bar{s} + H\sigma) + H\tau\mathbf{n}(\bar{s} + H\sigma), \quad \mathbf{r} \in \Omega_e, \tag{6.3}$$

where $\bar{s}$ is the arc length along $\partial\Omega$ to B and $\mathbf{n}$ is a unit *inward* normal to $\partial\Omega$. The assumption that $cH\kappa < 1$ at every point of $\partial\Omega$ insures that there is a 1:1 correspondence between the Cartesian and geodesic coordinates of any point in Ω_e.

If $\hat{\mathbf{r}}''(s)$ is continuous (as we shall henceforth assume), then Taylor's expansion with the remainder implies that

$$\hat{\mathbf{r}}(\bar{s} + H\sigma) = \bar{\mathbf{r}} + (H\sigma)\bar{\mathbf{t}} + \tfrac{1}{2}(H\sigma)^2\bar{\kappa}\bar{\mathbf{n}} + \mathrm{o}(H^2\sigma^2), \tag{6.4}$$

$$\mathbf{n}(\bar{s} + H\sigma) = \bar{\mathbf{n}} - (H\sigma)\bar{\kappa}\bar{\mathbf{t}} + \mathrm{o}(H\sigma), \tag{6.5}$$

where an overbar denotes a quantity evaluated at $\sigma = 0$. Thus, assuming $\sigma, \tau = \mathrm{O}(1)$, we have, from the first term in Eq. (4.5), and from Eqs. (6.3), and (6.4),

$$\boldsymbol{\rho}(\sigma, \tau) = \sigma\bar{\mathbf{t}} + \tau\bar{\mathbf{n}} + \varepsilon(\tfrac{1}{2}\sigma\bar{\mathbf{n}} - \tau\bar{\mathbf{t}})\bar{\kappa}L\sigma + \mathrm{o}(\varepsilon) \tag{6.6}$$

from which it follows that

$$\rho\cos\theta = \boldsymbol{\rho}\cdot\bar{\mathbf{t}} = \sigma - \varepsilon\bar{\kappa}L\sigma\tau + \mathrm{o}(\varepsilon), \tag{6.7}$$

$$\rho\sin\theta = \boldsymbol{\rho}\cdot\bar{\mathbf{n}} = \tau + \tfrac{1}{2}\varepsilon\bar{\kappa}L\sigma^2 + \mathrm{o}(\varepsilon). \tag{6.8}$$

Inverting these relations, we obtain

$$\sigma = \rho\cos\theta + \varepsilon\bar{\kappa}L\rho^2\sin\theta\cos\theta + \mathrm{o}(\varepsilon), \tag{6.9}$$

$$\tau = \rho\sin\theta - \tfrac{1}{2}\varepsilon\bar{\kappa}L\rho^2\cos^2\theta + \mathrm{o}(\varepsilon). \tag{6.10}$$

Finally, we note from Eqs. (5.6) and (6.6) that

$$\sigma_{\pm} = \pm\rho + \mathrm{o}(\varepsilon) \tag{6.11}$$

and from Eq. (6.10) that

$$\theta_{\pm} = \tfrac{1}{2}\varepsilon\bar{\kappa}L\rho + \mathrm{o}(\varepsilon). \tag{6.12}$$

As mentioned earlier, we expect that our unknowns in the edge zone Ω_e will vary slowly with σ, i.e., will vary slowly in a direction parallel to $\partial\Omega$. Thus, we assume that χ (which we have set to w outside Ω_e) has the form $\chi = \hat{\chi}(\varepsilon\sigma, \tau)$. We further assume that $\hat{\chi}$ has the expansion

$$\hat{\chi}(\varepsilon\sigma, \tau) = \chi_0(\tau) + \varepsilon\sigma\chi_1(\tau) + \mathrm{o}(\varepsilon). \tag{6.13}$$

By Eqs. (6.9) and (6.10), we can write Eq. (6.13) in the alternative form

$$\chi = \tilde{\chi}(\rho, \theta) = \chi_0(\rho\sin\theta) + \varepsilon\tilde{\chi}_1(\rho, \theta) + \mathrm{o}(\varepsilon), \tag{6.14}$$

where

$$\tilde{\chi}_1 = [\chi_1(\rho\sin\theta) - \tfrac{1}{2}\bar{\kappa}L(\rho\cos\theta)\chi_0'(\rho\sin\theta)]\rho\cos\theta. \tag{6.15}$$

Substituting Eq. (6.14) along with Eq. (6.12) into (5.12) and recalling that $\chi(\varepsilon\sigma, 0) = 0$, we have

$$\int_0^{\pi}\chi_0(\rho\sin\theta)\,\mathrm{d}\theta + \varepsilon\int_0^{\pi}\tilde{\chi}_1(\rho, \theta)\,\mathrm{d}\theta + \mathrm{o}(\varepsilon) = g(\rho), \tag{6.16}$$

which implies that

$$\int_0^{\pi} \chi_0(\rho \sin\theta)\,\mathrm{d}\theta = g(\rho) \tag{6.17}$$

and by Eq. (6.15) that

$$\chi_1(\rho \sin\theta) = \tfrac{1}{2}\overline{\kappa}L(\rho\cos\theta)\sigma\chi_0'(\rho\sin\theta). \tag{6.18}$$

To solve Eq. (6.17), let

$$\eta = \rho\sin\theta. \tag{6.19}$$

Then,

$$\int_0^{\rho} \frac{\chi_0(\eta)\,\mathrm{d}\eta}{\sqrt{\rho^2-\eta^2}} = \tfrac{1}{2}g(\rho). \tag{6.20}$$

As this equation is of the same form as Eq. (5.7), its solution is of the form as Eq. (5.10). That is,

$$\chi_0(\eta) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\int_0^{\eta}\frac{g(\xi)\,\mathrm{d}\xi}{\sqrt{\eta^2-\xi^2}}\right]. \tag{6.21}$$

7. Calculation of the residual loads and initial conditions: an outline

Our original goal, to compute surface shears q_x and q_y, residual body forces f_x, f_y, f_z, and initial conditions such that the three-dimensional stress and displacement fields inferred from Kirchhoff plate theory are exact solutions of elastodynamics, has been reduced to quadrature which, in general, requires numerics. (For the special case of the axisymmetric deformation of circular plates, analytic calculations are possible. See Qian and Simmonds (1998) for an example in statics and see Ladevèze and Simmonds (1996) for an example in the dynamics of beams.) Thus, we shall merely outline the steps necessary to compute these residual quantities. Note that all the computations are carried out with respect to some given point B on $\partial\Omega$, located by its arc length $\overline{s}$. Also, note that all computations assume that a solution w of the Kirchhoff plate equations is on hand

1. Compute $\tilde{f}$ from f, the right-hand side of Eq. (3.12), using Eqs. (A.1), (A.8), and (A.11) in Appendix A which express the modified transverse shear and normal stresses $\tilde{\tau}_x$, $\tilde{\tau}_y$, and $\overline{\sigma}$ in terms of w.
2. Compute $h(\overline{\zeta})$ from Eq. (5.9). If we locate the origin of our Cartesian coordinate system at the point B on $\partial\Omega$ so that $\mathbf{i} = \overline{\mathbf{t}}$ and $\mathbf{j} = \overline{\mathbf{n}}$, then from Eqs. (4.9), (5.6), (6.4), and (6.11), and recalling that we are suppressing the time variable, we have

$$\sigma_-'(\rho)\tilde{\tau}_s(-\sigma_-(\rho),\zeta) + \sigma_+'(\rho)\tilde{\tau}_s(\sigma_+(\rho),\zeta) = -\tilde{\tau}_y(\overline{s},\overline{\zeta} - H\zeta/c)[1+\mathrm{o}(1)]. \tag{7.1}$$

Thus, noting Eq. (6.12), we may write Eq. (5.9) in the form

$$h(\overline{\zeta}) = -\int_0^{\overline{\zeta}}\int_{\rho}^{\overline{\zeta}}\left[\frac{\alpha\tilde{\tau}_y(\overline{s},\overline{\zeta}-H\zeta/c) + \int_0^{\pi}\tilde{f}(\rho,\theta,\zeta)\,\mathrm{d}\theta}{\sqrt{\zeta^2-\rho^2}}\right]\mathrm{d}\zeta\,\mathrm{d}\rho[1+\mathrm{o}(1)], \tag{7.2}$$

which, if we ignore the o(1) error term, may be an acceptable approximation for thin plates. (Note that if $\hat{\mathbf{r}}(s)$ has a continuous third derivative along $\partial\Omega$, then O(1) terms become $\mathrm{O}(\varepsilon)$ terms.)
3. Solve Eq. (5.12) – mostly likely numerically – or, for a thin plate, use Eq. (6.21) to compute to a first approximation χ, and hence, $\phi = (2/\alpha)(w-\chi)$.

4. Compute the residual surface shears $\overline{q}_x$ and $\overline{q}_y$ from Eq. (3.1).
5. Use Eq. (4.21) to compute $W = \tilde{W} + \frac{1}{2}\alpha\phi$ anywhere in the plate – Eq. (5.4) will do outside the edge zone.
6. Integrate the system Eqs. (3.7) and (3.8) to find U_x and U_y.
7. Compute the right-hand sides of Eqs. (3.4)–(3.6) to find the residual body forces $\overline{f}_x$, $\overline{f}_y$, and $\overline{f}_z$.
8. Use Eq. (2.10) to compute the residual initial conditions $\{U_x^0, \ldots, \dot{W}\}$.

8. Closing remarks

The degree of non-vanishing of the residual surface shears, body forces, and initial conditions constitute a collective measure of the accuracy of Kirchhoff plate theory in any specific problem. Although tedious, the computations involved are straightforward. Once implemented, we believe that they will prove to be a practical tool for assessing the accuracy of dynamic solutions of the Kirchhoff plate theory; time will tell.

Acknowledgements

The work of J.G. Simmonds was supported by the U.S. Army Research Office under Grant DAAH04-94-G0189.

Appendix A. Three-dimensional fields implied by Kirchhoff theory

Our reference three-dimensional elasticity problem, which Eq. (2.11) and its associated initial/boundary conditions approximate, has no body forces, no surface shears, and homogeneous initial conditions. To derive the corresponding classical plate equations, augmented by approximate three-dimensional fields for the transverse shearing and normal stresses – the *augmented plate equations* for short – we introduce the following scaled variables into Eqs. (3.4)–(3.8):

$$\begin{gathered} x = L\xi, \qquad y = L\eta, \qquad z = H\zeta, \qquad t = (L^2/H)\sqrt{\rho/E}\tau, \\ \tilde{W} = (p_0L^4/EH^3)v, \qquad (U_x, U_y) = (p_0L^3/EH^2)(u_\xi, u_\eta), \\ (\tilde{\tau}_x, \tilde{\tau}_y) = (p_0L/EH)(T_\xi, T_\eta), \qquad \overline{\sigma} = (p_0/E)\Sigma, \end{gathered} \tag{A.1}$$

where p_0 is some nominal external pressure. (The variables ζ and τ are different from those used in the main body of this article.) We then let $\varepsilon = H/L \to 0$. Recalling that $\overline{f}_x = \overline{f}_y = \overline{f}_z = \phi = 0$ in our reference problem, we obtain, reversing the order of the equations,

$$v_{,\zeta} = 0, \qquad u_{\xi,\zeta} = -v_{,\xi}\,, \qquad u_\eta, \zeta = -v_{,\eta}\,, \tag{A.2}$$

$$(1 - v^2)T_{\xi,\zeta} + u_{\xi,\xi\xi} + \tfrac{1}{2}(1 - v)u_{\xi,\eta\eta} + \tfrac{1}{2}(1 + v)u_{\eta,\xi\eta} = 0, \tag{A.3}$$

$$(1 - v^2)T_{\eta,\zeta} + u_{\eta,\eta\eta} + \tfrac{1}{2}(1 - v)u_{\eta,\xi\xi} + \tfrac{1}{2}(1 + v)u_{\xi,\xi\eta} = 0, \tag{A.4}$$

$$v_{,\tau\tau} - (T_{\xi,\xi} + T_{\eta,\eta}) - \Sigma_{,\zeta} = 0. \tag{A.5}$$

Integrating these equations in order and remembering the parity conditions (2.7), we obtain from the first term in Eq. (A.2)

$$v = \hat{w}(\xi, \eta, \tau) \tag{A.6}$$

and thence from the second and third terms in Eq. (A.2),

$$u_\xi = -\zeta\hat{w}_{,\xi}(\xi, \eta, \tau), \qquad u_\eta = -\zeta\hat{w}_{,\eta}(\xi, \eta, \tau). \tag{A.7}$$

Substituting these expressions into Eqs. (A.3) and (A.4), integrating to find T_ξ and T_η, and imposing the face conditions $T_\xi(\xi, \eta, \pm 1, \tau) = T_\eta(\xi, \eta, \pm 1, \tau) = 0$, we find that

$$(1 - v)^2 T_\xi = \tfrac{1}{2}(\zeta^2 - 1)\Delta\hat{w}_{,\xi}\,, \qquad (1 - v)^2 T_\eta = \tfrac{1}{2}(\zeta^2 - 1)\Delta\hat{w}_{,\eta}\,, \tag{A.8}$$

where $\Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$.

Finally, integrating Eq. (A.5) from 0 to ζ, we have

$$(1 - v)^2 \Sigma = \tfrac{1}{2}(\zeta - \zeta^3/3)\Delta\Delta\hat{w} + (1 - v^2)\zeta\hat{w}_{,\tau\tau}\,. \tag{A.9}$$

Application of the face boundary condition $\Sigma(\xi, \eta, \pm 1, \tau) = \pm\frac{1}{2}(p/p_0)$ yields

$$(2/3)(1 - v^2)^{-1}\Delta\Delta\hat{w} + 2\hat{w}_{,\tau\tau} = p/p_0, \tag{A.10}$$

a dimensionless form of the Kirchhoff plate Eq. (2.11). Solving Eq. (A.10) for $\Delta\Delta\hat{w}$ and substituting the result back into Eq. (A.9), we have

$$\Sigma = (1/4)\zeta[(3 - \zeta^2)(p/p_0) + 2(\zeta^2 - 1)\hat{w}_{,\tau\tau}]. \tag{A.11}$$

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